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Deformed Clifford $Cl_q(n|m)$ and orthosymplectic $U_q[osp(2n+1|2m)]$ superalgebras and their root of unity representations

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Abstract

It is shown that the Clifford superalgebra $Cl(n|m)$ generated by m pairs of Bose operators (odd elements) anticommuting with n pairs of Fermi operators (even elements) can be deformed to $Cl_q(n|m)$ such that the latter is a homomorphic image of the quantum superalgebra $U_q[osp(2n+1|2m)]$. The Fock space $F(n|m)$ of $Cl_q(n|m)$ is constructed. For q being a root of unity ($q = \exp(i\pi l/k)$) q-bosons (and q-fermions) are operators acting in a finite-dimensional subspace $F_{l/k}(n|m)$ of $F(n|m)$. Each $F_{l/k}(n|m)$ is turned through the above-mentioned homomorphism into an irreducible (root of unity) $U_q[osp(2n+1|2m)]$ module. For q being a primitive root of unity ($l = 1$) the corresponding representation is unitary. The module $F_{1/k}(n|m)$ is decomposed into a direct sum of irreducible $U_q[sl(m|n)]$ submodules. The matrix elements of all Cartan–Weyl elements of $U_q[sl(m|n)]$ are given within each such submodule.

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1. Introduction

1.1. Some definitions

In this subsection, we give definitions which we need to present the motivations and the aims of the paper.

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Let (a_{ij}) [1] be an $(m+n)\times(m+n)$ symmetric Cartan matrix with entries

$$(a_{ij}) = (-1)^{\langle j\rangle}\delta_{i+1,j} + (-1)^{\langle i\rangle}\delta_{i,j+1} - [(-1)^{\langle j+1\rangle} + (-1)^{\langle j\rangle}]\delta_{ij} + \delta_{i,m+n}\delta_{j,m+n} \tag{1}$$

where

$$\langle i\rangle = \begin{cases}\bar{1} & \text{for} \quad i \leqslant m \\ \bar{0} & \text{for} \quad i > m\end{cases} \tag{2}$$

and $\mathbf{Z}_2 = \{\bar{0}, \bar{1}\}$ is the ring of all integers modulo 2.

Definition 1 ([1]). *$U_q[osp(2n+1|2m)]$ is a Hopf algebra, which is a topologically free $\mathbf{C}[[h]]$ module, $q = e^h$, (complete in the h-adic topology), with Chevalley generators h_i, e_i, f_i, $i = 1, \ldots, n+m = N$, subject to the following relations:*

(1) Cartan–Kac relations

$$[h_i, h_j] = 0 \tag{3a}$$

$$[h_i, e_j] = a_{ij}e_j \tag{3b}$$

$$[h_i, f_j] = -a_{ij}f_j \tag{3c}$$

$$[\![e_i, f_j]\!] = \delta_{ij}\frac{k_i - \bar{k}_i}{q - \bar{q}} \qquad q = e^h \quad \bar{q} = q^{-1} \quad k_i = q^{h_i} \quad \bar{k}_i = k_i^{-1} = q^{-h_i}. \tag{3d}$$

(2) e-Serre relations

$$[\![e_i, e_j]\!] = 0 \quad |i-j| \neq 1 \qquad e_m^2 = 0 \tag{4a}$$

$$[e_i, [e_i, e_{i\pm1}]_{\bar{q}}]_q \equiv [e_i, [e_i, e_{i\pm1}]_q]_{\bar{q}} = 0 \qquad i \neq m \quad i \neq m+n \tag{4b}$$

$$\{[e_m, e_{m-1}]_q, [e_m, e_{m+1}]_{\bar{q}}\} \equiv \{[e_m, e_{m-1}]_{\bar{q}}, [e_m, e_{m+1}]_q\} = 0 \tag{4c}$$

$$[e_N, [\![e_N, [e_N, e_{N-1}]_{\bar{q}}]\!]]_q \equiv [e_N, [\![e_N, [e_N, e_{N-1}]_q]\!]]_{\bar{q}} = 0. \tag{4d}$$

(3) f-Serre relations, obtained from (4) by replacing everywhere e_k with f_k

$$\begin{aligned}
&[\![f_i, f_j]\!] = 0 \quad |i-j| \neq 1 \qquad f_m^2 = 0 \\
&[f_i, [f_i, f_{i\pm1}]_{\bar{q}}]_q \equiv [f_i, [f_i, f_{i\pm1}]_q]_{\bar{q}} = 0 \qquad i \neq m \quad i \neq m+n \\
&\{[f_m, f_{m-1}]_q, [f_m, f_{m+1}]_{\bar{q}}\} \equiv \{[f_m, f_{m-1}]_q, [f_m, f_{m+1}]_q\} = 0 \\
&[f_N, [\![f_N, [f_N, f_{N-1}]_{\bar{q}}]\!]]_q \equiv [f_N, [\![f_N, [f_N, f_{N-1}]_q]\!]]_{\bar{q}} = 0.
\end{aligned} \tag{5}$$

The grading is induced from

$$\deg(h_j) = \bar{0} \qquad \forall j \qquad \deg(e_m) = \deg(f_m) = \bar{1} \quad \deg(e_i) = \deg(f_i) = \bar{0} \quad \text{for} \quad i \neq m.$$

Above and throughout, $\mathbf{C}[[h]]$ is the ring of all complex formal power series in h.

$$[a, b] = ab - ba \qquad \{a, b\} = ab + ba \qquad [\![a, b]\!] = ab - (-1)^{\deg(a)\deg(b)}ba \tag{6}$$

$$[a, b]_x = ab - xba \qquad \{a, b\}_x = ab + xba \qquad [\![a, b]\!]_x = ab - (-1)^{\deg(a)\deg(b)}xba. \tag{7}$$

We do not write the other Hopf superalgebra maps (comultiplication, co-unit and antipode) since we will not use them. Certainly, they are also a part of the definition.

An alternative description of the algebra under consideration in terms of the so-called deformed Green generators was given in [2]:

Definition 2. *$U_q[osp(2n+1|2m)]$ is a topologically free* $\mathbf{C}[[h]]$ *module ($q = e^h$) and an associative unital superalgebra with generators $H_i, a_i^{\pm}, i = 1, \ldots, m+n = N$ and relations*

$$[H_i, H_j] = 0 \tag{8a}$$

$$\left[H_i, a_j^{\pm}\right] = \pm\delta_{ij}(-1)^{\langle i\rangle} a_j^{\pm} \tag{8b}$$

$$\left[\!\left[a_i^-, a_i^+\right]\!\right] = -2\frac{L_i - \bar{L}_i}{q - \bar{q}} \qquad L_i = q^{H_i} \quad \bar{L}_i = q^{-H_i} \tag{8c}$$

$$\left[\left[\!\left[a_{N-1}^{\xi}, a_N^{\xi}\right]\!\right], a_N^{\xi}\right]_{q^{-(-1)^{\langle N\rangle}}} = 0 \tag{8d}$$

$$\left[\!\left[\left[\!\left[a_i^{\eta}, a_{i+\xi}^{-\eta}\right]\!\right], a_j^{\eta}\right]\!\right]_{q^{-\xi(-1)^{\langle i\rangle}\delta_{ij}}} = 2(\eta)^{\langle j\rangle}\delta_{j,i+\xi} L_j^{-\xi\eta} a_i^{\eta} \tag{8e}$$

where

$$\deg\left(a_i^{\pm}\right) \equiv \langle i\rangle. \tag{9}$$

The expressions of the deformed Green generators via the Chevalley generators read $(i = 1, \ldots, N-1)$

$$a_i^- = (-1)^{(m-i)\langle i\rangle}\sqrt{2}[e_i, [e_{i+1}, [\ldots, [e_{N-2}, [e_{N-1}, e_N]_{q_{N-1}}]_{q_{N-2}}\ldots]_{q_{i+2}}]_{q_{i+1}}]_{q_i}$$

$$a_N^- = \sqrt{2}e_N \tag{10a}$$

$$a_i^+ = (-1)^{N-i+1}\sqrt{2}[[[\ldots[f_N, f_{N-1}]_{\bar{q}_{N-1}}, f_{N-2}]_{\bar{q}_{N-2}}\ldots]_{\bar{q}_{i+2}}, f_{i+1}]_{\bar{q}_{i+1}}, f_i]_{\bar{q}_i}$$

$$a_N^+ = -\sqrt{2}f_N \tag{10b}$$

$$H_i = h_i + h_{i+1} + \cdots + h_N \qquad \textit{including}\, i = N \tag{10c}$$

with

$$q_i = q^{(-1)^{\langle i+1\rangle}} \qquad \text{i.e.} \quad q_i = \bar{q}, i < m \quad q_i = q, i \geqslant m. \tag{10d}$$

For the inverse relations, one finds

$$e_i = \frac{1}{2}\bar{L}_{i+1}\left[\!\left[a_i^-, a_{i+1}^+\right]\!\right] \qquad e_N = \frac{1}{\sqrt{2}}a_N^- \tag{10e}$$

$$f_i = -\frac{1}{2}(-1)^{\langle i+1\rangle}\left[\!\left[a_{i+1}^-, a_i^+\right]\!\right]L_{i+1} = \frac{1}{2}\left[\!\left[a_i^+, a_{i+1}^-\right]\!\right]L_{i+1} \qquad f_N = -\frac{1}{\sqrt{2}}a_N^+ \tag{10f}$$

$$h_i = H_i - H_{i+1} \qquad H_N = h_N. \tag{10g}$$

In the above definitions h is an abstract indeterminate. We shall be dealing however with representations of $U_q[osp(2n+1|2m)]$ where h is a complex number. The same also holds for the Clifford superalgebra defined in section 2 and for $U_q[sl(m|n)]$.

1.2. Motivations and aims

The motivation for giving a definition of $U_q[osp(2n+1|2m)]$ in terms of the deformed Green generators (definition 2) stems from the observation that the nondeformed Green generators $\hat{a}_i^{\pm}, i = 1, \ldots, m+n$, of $osp(2n+1|2m)$, contrary to the Chevalley generators, have a physical meaning [3]: $\hat{a}_1^{\pm}, \ldots, \hat{a}_m^{\pm}$ (respectively $\hat{a}_{m+1}^{\pm}, \ldots, \hat{a}_{m+n}^{\pm}$) are para-Bose (pB) (respectively para-Fermi (pF)) operators [4]. In one particular representation of $osp(2n+1|2m)$ the pB (respectively pF) operators become usual Bose (respectively Fermi) operators. Moreover, the grading of these operators is opposite to the canonical one: the Bose operators ($\equiv$ the bosons) are odd (fermionic) variables, the Fermi operators ($\equiv$ the fermions) are even (bosonic) variables, and the bosons anticommute with the fermions. Let us underline that the above somewhat unusual properties are not an input. On the contrary, they are an output from the representation theory of $osp(2n+1|2m)$.

The above result can also be viewed in the following way. Let $Cl(n|m)$ be the associative superalgebra which is generated by (i) m pairs of Bose operators $\hat{c}_1^{\pm}, \ldots, \hat{c}_m^{\pm}$ (postulated to be odd variables), (ii) n pairs of Fermi operators $\hat{c}_{m+1}^{\pm}, \ldots, \hat{c}_{m+n}^{\pm}$ (even elements) and (iii) the assumption that the bosons anticommute with the fermions. Then the linear map φ defined on the generators as

$$\varphi(\hat{a}_i^{\pm}) = \hat{c}_i^{\pm} \qquad i = 1, \ldots, n+m \tag{11}$$

and extended by associativity, is a homomorphism of the universal enveloping algebra (UEA) $U[osp(2n+1|2m)]$ onto $Cl(n|m)$.

Independently from [3], bosons and fermions with the above properties (in the case $n = m = \infty$) were used in [5] for studying the irreducible representations of some infinite-dimensional superalgebras (including $osp(\infty|\infty)$ and $gl(\infty|\infty)$), relevant for construction of supersymmetric generalizations of certain hierarchies of soliton-like evolution equations. Accepting the terminology of this paper, we refer to $Cl(n|m)$ as Clifford superalgebra.

There are different approaches for extending the results of single-mode deformed bosons $b^{\pm}$ [6–8] or fermions $f^{\pm}$ [9] to the multi-mode case $b_i^{\pm}, f_i^{\pm}, i = 1, 2, \ldots$. One way is to preserve the main features of the nondeformed case, postulating that different modes of q-bosons commute and different modes of q-fermions anticommute (see [9] for the precise definitions). In other approaches, the relations between different modes are not postulated. They are derived on the ground of other assumptions, yielding as a rule that different modes of operators do not commute, but *q-commute* [10–13].

Our approach for (simultaneous) deformation of bosons and fermions is based on the requirement that equation (11) be preserved also in the quantum case. More precisely, we require that the relations between the generators of the deformed Clifford superalgebra $Cl_q(n|m)$, namely the relations between m pairs of deformed bosons (q-bosons) $c_1^{\pm}, \ldots, c_m^{\pm}$ and n pairs of deformed fermions (q-fermions) $c_{m+1}^{\pm}, \ldots, c_{m+n}^{\pm}$, are determined in such a way that the q-analogue of the above map (11) is a homomorphism of $U_q[osp(2n+1|2m)]$ onto $Cl_q(n|m)$. As a result, the single-mode q-bosons (respectively q-fermions) coincide (up to an overall multiple) with the known q-bosons [6–8, 14], but their different modes 'q-commute' (respectively 'q-anticommute'). The deformed bosons and fermions mutually q-anticommute. The q-bosons $c_1^{\pm}, \ldots, c_m^{\pm}$ coincide with those introduced in [15] in connection with the quantum superalgebra $U_q[osp(1|2n)]$.

One of the consequences of the present approach stems from the observation that any (irreducible) $Cl_q(n|m)$ module V and in particular its Fock space $F(n|m)$ can be immediately turned into a (irreducible) representation space of $U_q[osp(2n+1|2m)]$ (and also into a representation space of any subalgebra of $U_q[osp(2n+1|2m)]$, in particular of $U_q[sl(m|n)]$. Based on this, we shall construct root of unity irreducible representations of $U_q[osp(2n+1|2m)]$ and of $U_q[sl(m|n)]$ in (appropriate subspaces/factorspaces of) $F(n|m)$.

In section 2 we introduce the deformed Clifford algebra $Cl_q(n|m)$ and study its Fock space $F(n|m)$. We write the transformations of the basis under the action of the $Cl_q(n|m)$ generators (equations (15)). Next we show that when $q = \exp(i\pi l/k)$ with l, k integers, i.e. when q is a root of unity, $F(n|m)$ contains an infinite-dimensional invariant (and depending on q) subspace $I_{l/k}$, so that the factor space $F_{l/k}(n|m) = F(n|m)/I_{l/k}$ is a finite-dimensional irreducible $Cl_q(n|m)$ module. We show that the representation of $Cl_q(n|m)$ in $F_{l/k}(n|m)$ is unitary (with respect to a natural anti-involution) only if q is a primitive root of unity ($l = 1$). We write explicitly the transformations of the basis of $F_{1/k}(n|m)$ under the action of the deformed Bose and Fermi operators (equations (46)).

Using the homomorphism φ of $U_q[osp(2n+1|2m)]$ onto $Cl_q(n|m)$ (see proposition 6), in section 3 we turn each Fock space $F_{l/k}(n|m)$ into an irreducible $U_q[osp(2n+1|2m)]$ module, thus finding a class of representations of this quantum superalgebra. Next we consider $U_q[sl(m|n)]$ as a subalgebra of $U_q[osp(2n+1|2m)]$. As one can expect, the Fock spaces carry reducible representations of $U_q[sl(m|n)]$. Each Fock space $F_{1/k}(n|m)$ is decomposed into a direct sum of $U_q[sl(m|n)]$ irreducible subspaces. We write the transformations of these irreducible $U_q[sl(m|n)]$ modules not only with respect to the Chevalley generators but also under the action of all Cartan–Weyl elements of $U_q[sl(m|n)]$.

2. A q-deformed Clifford superalgebra $Cl_q(n|m)$ and its Fock representations

In this section, we define a q-deformed Clifford superalgebra $Cl_q(n|m)$, generated by a set of operators interpreted as deformed Bose and Fermi operators. We study a class of Fock representations of $Cl_q(n|m)$, which are finite dimensional in the case when q is a root of unity. In this way parallel to the q-fermions also the q-bosons are finite dimensional.

The restriction that q be a root of unity is not artificial. It is a consequence of basic physical requirements (e.g. the physical observables have to be Hermitian operators). Mathematically, this leads to representations which are unitary with respect to a natural antilinear anti-involution (see equation (21)). It turns out that only the representations corresponding to q being a primitive root of unity are unitary.

The results of the present section will be used in the next section in order to define root of unity representations of the superalgebra $U_q[osp(2n+1|2m)]$ and its subalgebra $U_q[sl(m|n)]$.

Define a set of $\mathbf{Z}_2$-graded operators $c_i^\pm$, N_i,

$$\deg\left(c_i^\pm\right) = \langle i\rangle \qquad \deg(N_i) = \bar{0} \qquad i = 1, \ldots, m+n$$

with relations

$$[N_i, N_j] = 0 \tag{12a}$$

$$\left[N_i, c_j^\pm\right] = \pm\delta_{ij} c_j^\pm \tag{12b}$$

$$c_i^- c_i^+ + (-1)^{\langle i\rangle} q^{\pm 1} c_i^+ c_i^- = \frac{2}{q^{1/2} + q^{-1/2}} q^{\pm(-1)^{\langle i\rangle} N_i} \tag{12c}$$

$$c_i^\xi c_j^\eta = -(-1)^{\langle j\rangle} q^{\xi\eta} c_j^\eta c_i^\xi \qquad \text{for all} \quad i < j \quad \xi, \eta = \pm \text{ or } \pm 1 \tag{12d}$$

$$\left(c_i^\pm\right)^2 = 0 \qquad i = m+1, \ldots, m+n. \tag{12e}$$

Note that equations (12c) are equivalent to

$$c_i^+ c_i^- = c_q \frac{q^{N_i} - \bar{q}^{N_i}}{q - \bar{q}} \qquad c_q = \frac{2}{q^{1/2} + \bar{q}^{1/2}} \tag{12f}$$

$$c_i^- c_i^+ = c_q \frac{q^{1-(-1)^{\langle i\rangle} N_i} - \bar{q}^{1-(-1)^{\langle i\rangle} N_i}}{q - \bar{q}}. \tag{12g}$$

It is obvious that in the limit $q \to 1$ the operators $c_1^\pm, \ldots, c_m^\pm$ (respectively $c_{m+1}^\pm, \ldots, c_{m+n}^\pm$) reduce to Bose (respectively Fermi) creation and annihilation operators, which mutually anticommute, and N_i are the corresponding number operators.

Definition 3. *The deformed Clifford algebra $Cl_q(n|m)$ is a topologically free* $\mathbf{C}[[h]]$ *module and an associative unital superalgebra with generators $c_i^{\pm}$, N_i subject to relations (12).*

Clearly, $Cl_q(n|m)$ is a deformation of the Clifford superalgebra $Cl(n|m)$.

We proceed with the construction of the Fock space $F(n|m)$ of $Cl_q(n|m)$. The vacuum vector $|0\rangle$ is defined in a natural way:

$$c_i^-|0\rangle = 0 \quad N_i|0\rangle = |0\rangle \qquad i = 1, \ldots, m+n. \tag{13}$$

As a basis, take

$$\begin{aligned}&\left(c_1^+\right)^{r_1}\left(c_2^+\right)^{r_2}\ldots\left(c_{m+n}^+\right)^{r_{m+n}}|0\rangle = |r_1, r_2, \ldots, r_{m+n}\rangle \qquad r_i \in \mathbf{Z}_+ \quad i = 1, \ldots, m\\ &\qquad r_i \in \{0, 1\} \quad i = m+1, \ldots, m+n\end{aligned} \tag{14}$$

where $\mathbf{Z}_+$ are all non-negative integers.

Proposition 1. *The transformation of the basis (14) under the action of the deformed Bose and Fermi operators reads*

$$N_i|r_1, \ldots, r_{m+n}\rangle = r_i|r_1, \ldots, r_{m+n}\rangle \tag{15a}$$

$$c_i^+|r_1, \ldots, r_{m+n}\rangle = (-1)^{(1-\langle i\rangle)(r_1+\cdots+r_{i-1})}(1-(1-\langle i\rangle)r_i)\bar{q}^{r_1+\cdots+r_{i-1}}|\ldots, r_{i-1}, r_i+1, r_{i+1}, \ldots\rangle \tag{15b}$$

$$c_i^-|r_1, \ldots, r_{m+n}\rangle = (-1)^{(1-\langle i\rangle)(r_1+\cdots+r_{i-1})}[r_i]c_q q^{r_1+\cdots+r_{i-1}}|\ldots, r_{i-1}, r_i-1, r_{i+1}, \ldots\rangle \tag{15c}$$

where

$$[x] = \frac{q^x - q^{-x}}{q - \bar{q}}. \tag{16}$$

Proof. The proof follows from the defining relations (12) and identities such as

$$1 + q^2 + q^4 + \cdots + q^{2n} + \bar{q}^2 + \bar{q}^4 + \cdots + \bar{q}^{2n} = [2n+1] \tag{17}$$

$$q + q^3 + \cdots + q^{2n+1} + \bar{q} + \bar{q}^3 + \cdots + \bar{q}^{2n+1} = [2n+2] \tag{18}$$

$$c_i^-\left(c_i^+\right)^n = \bar{q}^n\left(c_i^+\right)^n c_i^- + c_q[n]\left(c_i^+\right)^{n-1}q^{N_i} \qquad i = 1, \ldots, m. \tag{19}$$

□

On the ground of the Fock space $F(n|m)$ we proceed to determine a class of unitary representations [16] of $Cl_q(n|m)$ with respect to the antilinear anti-involution ω defined by

$$\omega\left(c_i^{\pm}\right) = c_i^{\mp} \qquad \omega(N_i) = N_i. \tag{20}$$

If (,) is a nondegenerate Hermitian form then (20) is equivalent to

$$\left(c_i^+\right)^\dagger = c_i^- \quad (N_i)^\dagger = N_i \qquad i = 1, \ldots, m+n \tag{21}$$

with $x^\dagger$ being the conjugate to x operator with respect to (,) and the corresponding representation is said to be contravariant. If (,) is a scalar product, the representation is unitary.

The unitarity condition (21) stems from physical considerations. For example, in case of a harmonic oscillator it is convenient to replace the position operators q_k and the momentum p_k via creation and annihilation operators $c_k^{\pm}$: $q_k = \left(c_k^+ + c_k^-\right)/\sqrt{2}$, $p_k = \mathrm{i}\left(c_k^+ - c_k^-\right)/\sqrt{2}$.

Then the necessity that q_k and p_k be Hermitian operators leads to equation (21). For the same reason, the unitarity condition has to hold in any second quantized picture, in the filling number representation, etc. Let us add that condition (21) is statistically independent; it has to hold for any statistics compatible with the principles of quantum theory.

Define a Hermitian form (,) on $F(n|m)$ requiring

$$(|0\rangle, |0\rangle) = \langle 0|0\rangle = 1 \tag{22}$$

and postulating that (21) should be satisfied, namely

$$\left(c_i^{\pm} v, w\right) = \left(v, c_i^{\mp} w\right) \quad (N_i v, w) = (v, N_i w) \qquad v, w \in F(n|m). \tag{23}$$

Proposition 2. *The above definition is self-consistent only if q is a pure phase, more precisely if*

$$q = \mathrm{e}^{\mathrm{i}\phi} \qquad 0 \leqslant \phi < 2\pi \quad \phi \neq \pi. \tag{24}$$

Proof. Only for simplicity we assume that $m > 1$. From (12), (13), (22) and (23) one derives uniquely that

$$(|1, \dot{0}\rangle, |1, \dot{0}\rangle) = (|0, 1, \dot{0}\rangle, |0, 1, \dot{0}\rangle) = c_q \tag{25}$$

where $\dot{0}$ indicates that all remaining entries are zero, $|1, \dot{0}\rangle = |1, 0, \ldots, 0\rangle$. c_q has to be a nonzero real number. Compute now $(|1, 1, \dot{0}\rangle, |1, 1, \dot{0}\rangle)$:

$$\begin{aligned}(|1, 1, \dot{0}\rangle, |1, 1, \dot{0}\rangle) &= \left(c_1^+ c_2^+ |0\rangle, c_1^+ c_2^+ |0\rangle\right) = \left(c_2^+ |0\rangle, c_1^- c_1^+ c_2^+ |0\rangle\right) \\ &= \left(c_2^+|0\rangle, c_q \frac{q^{1+N_1} - \bar{q}^{1+N_1}}{q - \bar{q}} c_2^+ |0\rangle\right) = c_q (|0, 1, \dot{0}\rangle, |0, 1, \dot{0}\rangle) = c_q^2. \end{aligned} \tag{26}$$

On the other hand, the LHS of (26) can be evaluated in a different way:

$$\begin{aligned}(|1, 1, \dot{0}\rangle, |1, 1, \dot{0}\rangle) &= \left(c_1^+ c_2^+ |0\rangle, c_1^+ c_2^+ |0\rangle\right) = \left(q c_2^+ c_1^+ |0\rangle, q c_2^+ c_1^+ |0\rangle\right) = |q|^2 \left(c_1^+|0\rangle, c_2^- c_2^+ c_1^+ |0\rangle\right) \\ &= |q|^2 \left(c_1^+|0\rangle, c_q \frac{q^{1+N_2} - \bar{q}^{1+N_2}}{q - \bar{q}} c_1^+ |0\rangle\right) = |q|^2 c_q (|1, \dot{0}\rangle, |1, \dot{0}\rangle) = |q|^2 c_q^2. \end{aligned} \tag{27}$$

Equations (26) and (27) are compatible only for $|q| = 1$. From the latter and (12*f*) it follows that $c_q = (\cos(\phi/2))^{-1}$ and therefore $\phi \neq \pi$. Hence (24) holds. □

Any two vectors $|r_1, \ldots, r_{m+n}\rangle$ and $|r_1', \ldots, r_{m+n}'\rangle$ with $(r_1, \ldots, r_{m+n}) \neq (r_1', \ldots, r_{m+n}')$ are orthogonal and

$$(|r_1, \ldots, r_{m+n}\rangle, |r_1, \ldots, r_{m+n}\rangle) = c_q^{r_1+\cdots+r_{m+n}} [r_1]![r_2]! \ldots [r_m]! \tag{28}$$

where $[r]! = [r][r-1] \ldots [1]$. Hence (,) is a nondegenerate Hermitian form and therefore equation (21) is a consequence of (20).

Proposition 3. *The Hermitian form (,) is not positive definite on $F(n|m)$.*

Proof. Consider a vector $|r, \dot{0}\rangle$. We recall, see (24), that $0 \leqslant \phi < 2\pi$, $\phi \neq \pi$. For $\phi = 0$ the RHS of (28) yields $(|r, \dot{0}\rangle, |r, \dot{0}\rangle) = r! > 0$.

If $\phi \neq 0$ (and $\phi \neq \pi$) (28) can be written as

$$(|r, \dot{0}\rangle, |r, \dot{0}\rangle) = (\cos(\phi/2))^{-r} (\sin(\phi))^{-r} \sin(\phi) \sin(2\phi) \sin(3\phi) \ldots \sin(r\phi). \tag{29}$$

For $0 < \phi < \pi$ both $(\cos(\phi/2))^{-r} > 0$ and $(\sin(\phi))^{-r} > 0$. Let k be the smallest positive integer such that $0 < k\phi < \pi$ and $\pi < (k+1)\phi < 2\pi$. Then $(|k, \dot{0}\rangle, |k, \dot{0}\rangle) > 0$, but

$$(|k+1, \dot{0}\rangle, |k+1, \dot{0}\rangle) = (|k, \dot{0}\rangle, |k, \dot{0}\rangle)(\cos(\phi/2))^{-1}(\sin(\phi))^{-1}\sin((k+1)\phi) < 0. \tag{30}$$

Since the replacement of ϕ with $-\phi$ does not change the RHS of (30), $(|k+1, \dot{0}\rangle, |k+1, \dot{0}\rangle) < 0$ also for $\pi < \phi < 2\pi$. This completes the proof. □

In view of the above proposition, the Hermitian form (,) could be positive definite eventually on $Cl_q(n|m)$-invariant subspaces of $F(n|m)$ or in their factor spaces. For q being a root of unity, $F(n|m)$ contains such invariant subspaces. Indeed, set

$$q = \exp(\mathrm{i}\pi l/k) \quad \Longleftrightarrow \quad \phi = \pi l/k \qquad \text{with } l \text{ and } k \text{ being relatively simple integers} \tag{31}$$

which (without loss of generality) are assumed to be positive integers and $l < k$. It is straightforward to verify that

$$I_{l/k}(n|m) = \mathrm{span}\{|r_1, \ldots, r_i, \ldots, r_{m+n}\rangle | r_1 \geqslant k, r_2 \geqslant k, \ldots, r_m \geqslant k\} \tag{32}$$

is an (infinite-dimensional) invariant subspace. The latter follows from (15c), namely

$$c_j^-|r_1, \ldots, r_j, \ldots, r_{m+n}\rangle = c_q[r_j]q^{r_1+\cdots+r_{j-1}}|\ldots, r_j - 1, \ldots\rangle \qquad j = 1, \ldots, m \tag{33}$$

and the circumstance that $[r_j] = 0$ for $r_j = k$ (and $q = \exp(\mathrm{i}\pi l/k)$).

Let $F_{l/k}(n|m)$ be the factor space of $F(n|m)$ with respect to $I_{l/k}(n|m)$,

$$F_{l/k}(n|m) = F(n|m)/I_{l/k}(n|m). \tag{34}$$

All vectors (considered as representatives of the equivalence classes),

$$|r_1, \ldots, r_{m+n}\rangle \qquad 0 \leqslant r_1, \ldots, r_m \leqslant k-1,\ 0 \leqslant r_{m+1}, \ldots, r_{m+n} \leqslant 1 \tag{35}$$

constitute a basis in $F_{l/k}(n|m)$. For this reason, we state that $F_{l/k}(n|m)$ can be considered as a subspace of $F(n|m)$.

Proposition 4. *The Fock space $F_{l/k}(n|m)$ is a finite-dimensional irreducible $Cl_q(n|m)$ module.*

Proof.

(i) Let $j = 1, \ldots, m$. Than (15c) yields

$$c_j^-|r_1, \ldots, r_j, \ldots, r_{m+n}\rangle = \exp\left(\mathrm{i}\pi\frac{l}{k}(r_1 + \cdots + r_{j-1})\right)\cos(\pi l/2k)\frac{\sin\left(\pi\frac{l}{k}r_j\right)}{\sin\left(\pi\frac{l}{k}\right)} \times |\ldots, r_j - 1, \ldots\rangle. \tag{36}$$

Clearly $(-1)\exp\left(\mathrm{i}\pi\frac{l}{k}(r_1 + \cdots + r_{j-1})\right) \neq 0$. Since $l < k \in \mathbf{N}$ ($\mathbf{N}$—all positive integers) $\cos(\pi l/2k) > 0$ and $\sin\left(\pi\frac{l}{k}\right) > 0$. The next observation is that $\frac{l}{k}r_j$ is not an integer for $0 < r_j < k$. Indeed, suppose that $\frac{l}{k}r_j$ is an integer and let s be the maximal integer common divisor of k and r_j, i.e. $k = s.k'$ and $r_j = s.r'$. Then $\frac{l}{k}r_j = \frac{l.r'}{k'}$ and the integers $l.r'$ and k' have no common divisors, i.e. $\frac{l}{k}r_j$ is not an integer. Therefore $\sin\left(\pi\frac{l}{k}r_j\right) \neq 0$ and as a result the coefficient in front of $|\ldots, r_j - 1, \ldots\rangle$ in the RHS of (36) is different from zero for any r_j, $0 < r_j < k$.

(ii) If $m < j < m+n$ and $r_j = 1$, then

$$c_j^- |\ldots, r_j = 1, \ldots\rangle = (-1)^{r_1+\cdots+r_{j-1}} \exp\left(\mathrm{i}\pi \frac{l}{k}(r_1 + \cdots + r_{j-1})\right) \cos(\pi l/2k) |\ldots, r_j = 0, \ldots\rangle \tag{37}$$

and again the RHS of (37) is different from zero.

As a result

$$(c_1^-)^{r_1} \ldots (c_{m+n}^-)^{r_{m+n}} |r_1, \ldots, r_{m+n}\rangle = \text{const}\,|0\rangle \qquad \text{const} \neq 0. \tag{38}$$

From (38) and (15*b*) one easily concludes that $F_{l/k}(n|m)$ is an irreducible $Cl_q(n|m)$ module. □

So far we have defined a class of contravariant representations. The problem that still remains to be solved is to select out of them those $F_{l/k}(n|m)$ modules for which the metric (,) is positive definite, namely the modules which carry unitary representations.

Proposition 5. *The Fock space $F_{l/k}(n|m)$ is a Hilbert space only if $l = 1$ (and for any $1 < k \in \mathbf{N}$), i.e. if q is a primitive root of unity,*

$$q = \exp(\mathrm{i}\pi/k) \iff \phi = \frac{\pi}{k}. \tag{39}$$

Proof.

(i) Assume $1 < l < k$.

For $q = \exp(\mathrm{i}\pi l/k)$ and $1 \leqslant r < k$ one obtains from (28) that

$$\begin{aligned}(|r, \dot{0}\rangle, |r, \dot{0}\rangle) = {} & \left(\cos\left(\frac{\pi}{2}\frac{l}{k}\right)\right)^{-r} \left(\sin\left(\pi\frac{l}{k}\right)\right)^{-r} \sin\left(\pi\frac{l}{k}1\right) \\ & \times \sin\left(\pi\frac{l}{k}2\right) \sin\left(\pi\frac{l}{k}3\right) \ldots \sin\left(\pi\frac{l}{k}r\right).\end{aligned} \tag{40}$$

Clearly,

$$\cos\left(\frac{\pi}{2}\frac{l}{k}\right) > 0 \quad \sin\left(\pi\frac{l}{k}\right) > 0 \quad \text{whereas} \quad \sin\left(\pi\frac{l}{k}(k-1)\right) < 0. \tag{41}$$

Therefore there exists an integer r_0, $1 \leqslant r_0 < k-1$ such that

$$\pi\frac{l}{k}1 < \pi \quad \pi\frac{l}{k}2 < \pi, \ldots, \pi\frac{l}{k}r_0 < \pi \quad \text{whereas} \quad \pi\frac{l}{k}(r_0+1) > \pi. \tag{42}$$

Then (40) yields $(|r_0+1\rangle, |r_0+1\rangle) < 0$. This proves that the Hermitian form (,) is not a scalar product if $l \neq 1$ (if q is not a primitive root of unity).

(ii) If (39) holds, then (28) yields

$$(|r_1, \ldots, r_{m+n}\rangle, |r_1, \ldots, r_{m+n}\rangle) = \cos\left(\frac{\pi}{2k}\right)^{-(r_1+\cdots+r_{m+n})} [r_1]! \ldots [r_j]! \ldots [r_m]!. \tag{43}$$

For each $j = 1, \ldots, m$

$$[r_j]! = \prod_{s=1}^{r_j} \frac{\sin\left(\frac{\pi}{k}s\right)}{\sin\left(\frac{\pi}{k}\right)} > 0 \qquad \text{since} \quad s \leqslant r_j < k. \tag{44}$$

Moreover, since $\cos(\pi/2k) > 0$, the RHS of (43) is also positive. This completes the proof. □

Clearly, the dimension of the Fock space $F_{1/k}(n|m)$ is $k^m 2^n$. Define an orthonormal basis in $F_{1/k}(n|m)$,

$$|r_1,\ldots,r_{m+n}) = \left(c_q^{r_1+\cdots+r_{m+n}}[r_1]![r_2]!\ldots[r_m]!\right)^{-1/2}|r_1,\ldots,r_{m+n}\rangle. \tag{45}$$

The transformation of this basis reads ($1 < k \in \mathbf{N}$)

$$N_j|r_1,\ldots,r_{m+n}) = r_j|r_1,\ldots,r_{m+n}) \tag{46a}$$

$$c_j^+|r_1,\ldots,r_{m+n}) = \exp(-\mathrm{i}\pi(r_1+r_2+\cdots+r_{j-1})/k)\sqrt{\frac{2\sin(\pi(r_j+1)/k)\sin(\pi/(2k))}{\sin^2(\pi/k)}} \\ \times|r_1,\ldots,r_{j-1},r_j+1,r_{j+1},\ldots,r_{m+n}) \qquad j=1,\ldots,m \tag{46b}$$

$$c_j^-|r_1,\ldots,r_{m+n}) = \exp(\mathrm{i}\pi(r_1+r_2+\cdots+r_{j-1})/k)\sqrt{\frac{2\sin(\pi r_j/k)\sin(\pi/(2k))}{\sin^2(\pi/k)}} \\ \times|r_1,\ldots,r_{j-1},r_j-1,r_{j+1},\ldots,r_{m+n}) \qquad j=1,\ldots,m \tag{46c}$$

$$c_j^+|r_1,\ldots,r_{m+n}) = (-1)^{r_1+\cdots+r_{j-1}}(1-r_j)\exp(-\mathrm{i}\pi(r_1+\cdots+r_{j-1})/k)\sqrt{\frac{1}{\cos(\pi/(2k))}} \\ \times|r_1,\ldots,r_{j-1},r_j+1,r_{j+1},\ldots,r_{m+n}), \qquad j=m+1,\ldots,m+n \tag{46d}$$

$$c_j^-|r_1,\ldots,r_{m+n}) = (-1)^{r_1+\cdots+r_{j-1}}r_j\exp(\mathrm{i}\pi(r_1+\cdots+r_{j-1})/k)\sqrt{\frac{1}{\cos(\pi/(2k))}} \\ \times|r_1,\ldots,r_{j-1},r_j-1,r_{j+1},\ldots,r_{m+n}) \qquad j=m+1,\ldots,m+n. \tag{46e}$$

The above relations yield that in the root of unity case the bosons are 'finite dimensional', i.e. they are operators in finite-dimensional state spaces.

3. A homomorphism of $U_q[osp(2n+1|2m)]$ onto $Cl_q(n|m)$: applications to $U_q[osp(2n+1|2m)]$ and $U_q[sl(m|n)]$ representations

In what follows, we observe that the deformed Bose and Fermi operators (12) satisfy the defining relations (8) of the algebra $U_q[osp(2n+1|2m)]$. Therefore, relations (46) give a root of unity representation of $U_q[osp(2n+1|2m)]$ also. The Cartan–Weyl elements of the subalgebra $U_q[sl(m|n)] \subset U_q[osp(2n+1|2m)]$ are expressed in terms of the q-anticommuting deformed Bose and Fermi operators. The decomposition of $U_q[osp(2n+1|2m)]$ Fock space $F_{1/k}(n|m)$ with respect to $U_q[sl(m|n)]$ is considered, giving rise to root of unity representations for the latter also.

Proposition 6. *The linear map $\varphi : U_q[osp(2n+1|2m)] \rightarrow Cl_q(n|m)$, defined on the generators of $U_q[osp(2n+1|2m)]$ as*

$$\varphi\left(a_i^\pm\right) = c_i^\pm \quad \varphi(H_i) = (-1)^{\langle i\rangle}N_i - \tfrac{1}{2} \qquad i=1,\ldots,m+n \tag{47}$$

and extended on all elements by associativity is a homomorphism (in the sense of associative superalgebras) of $U_q[osp(2n+1|2m)]$ onto $Cl_q(n|m)$.

Proof. The proof follows from the fact that the images $\varphi(a_i^{\pm})$ and $\varphi(H_i)$ of the generators of $U_q[osp(2n+1|2m)]$ satisfy also the defining relations (8). Moreover, the generators of $Cl_q(n|m)$ are among the images of φ: $c_i^{\pm} = \varphi(a_i^{\pm})$, $N_i = \varphi\big((-1)^{\langle i\rangle}\big(H_i + \frac{1}{2}\big)\big)$. □

The relevance of proposition 6 stems from the observation that any $Cl_q(n|m)$ module F and in particular the Fock modules studied in the previous section can be turned into $U_q[osp(2n+1|2m)]$ modules simply by setting $\varphi(a)x$ for any $a \in U_q[osp(2n+1|2m)]$ and $x \in F$. The conclusion from (47) is that equations (46) define an irreducible $U_q[osp(2n+1|2m)]$ representation for any q being a primitive root of unity.

Next, we use the circumstance that the quantum superalgebra $U_q[sl(m|n)]$ is a subalgebra of $U_q[osp(2n+1|2m)]$. Therefore, each $U_q[osp(2n+1|2m)]$ module is (generally reducible) $U_q[sl(m|n)]$ module. Using this, we proceed to decompose each $U_q[osp(2n+1|2m)]$ module $F_{1/k}(n|m)$ into a direct sum of irreducible $U_q[sl(m|n)]$ modules, thus defining explicitly a class of root of unity representations of the quantum superalgebra $U_q[sl(m|n)]$.

As a first step, we recall the definition of the quantum superalgebra $U_q[sl(m|n)]$. Let (α_{ij}) be the $(m+n-1)\times(m+n-1)$ Cartan matrix with entries

$$\alpha_{ij} = (1+(-1)^{\theta_{i,i+1}})\delta_{ij} - (-1)^{\theta_{i,i+1}}\delta_{i,j-1} - \delta_{i-1,j} \qquad i,j \in [1; n+m-1] \tag{48}$$

$$\theta_i = 1 - \langle i\rangle = \begin{cases} \bar{0} & \text{if} \quad i \leqslant m \\ \bar{1}, & \text{if} \quad i > m \end{cases} \qquad \theta_{ij} = \theta_i + \theta_j. \tag{49}$$

Definition 4. *$U_q[sl(m|n)]$ is a Hopf algebra, which is a topologically free* $\mathbf{C}[[h]]$ *module, with Chevalley generators $\hat{h}_i, \hat{e}_i, \hat{f}_i$, $i = 1,\ldots,m+n-1$, subject to the following relations* $(\deg(\hat{h}_i) = \bar{0}$, $\deg(\hat{e}_i) = \deg(\hat{f}_i) = \theta_{i,i+1})$:

(1) Cartan–Kac relations

$$[\hat{h}_i, \hat{h}_j] = 0 \tag{50a}$$

$$[\hat{h}_i, \hat{e}_j] = \alpha_{ij}\hat{e}_j \qquad [\hat{h}_i, \hat{f}_j] = -\alpha_{ij}\hat{f}_j \tag{50b}$$

$$[\![\hat{e}_i, \hat{f}_j]\!] = \delta_{ij}\frac{\hat{k}_i - \bar{\hat{k}}_i}{q-\bar{q}} \qquad \hat{k}_i = q^{\hat{h}_i} \quad \hat{k}_i^{-1} \equiv \bar{\hat{k}}_i = q^{-\hat{h}_i}. \tag{50c}$$

(2) $\hat{e}$-Serre relations

$$[\hat{e}_i, \hat{e}_j] = 0 \quad \textit{if} \quad |i-j| \neq 1 \qquad \hat{e}_m^2 = 0 \tag{51a}$$

$$[\hat{e}_i, [\hat{e}_i, \hat{e}_{i\pm1}]_{\bar{q}}]_q = [\hat{e}_i, [\hat{e}_i, \hat{e}_{i\pm1}]_q]_{\bar{q}} = 0 \qquad i \neq m \tag{51b}$$

$$\{\hat{e}_m, [[\hat{e}_{m-1}, \hat{e}_m]_q, \hat{e}_{m+1}]_{\bar{q}}\} = \{\hat{e}_m, [[\hat{e}_{m-1}, \hat{e}_m]_{\bar{q}}, \hat{e}_{m+1}]_q\} = 0. \tag{51c}$$

(3) $\hat{f}$-Serre relations,

$$\begin{gathered}
[\hat{f}_i, \hat{f}_j] = 0 \quad \textit{if} \quad |i-j| \neq 1 \qquad \hat{f}_m^2 = 0 \\
[\hat{f}_i, [\hat{f}_i, \hat{f}_{i\pm1}]_{\bar{q}}]_q = [\hat{f}_i, [\hat{f}_i, \hat{f}_{i\pm1}]_q]_{\bar{q}} = 0 \qquad i \neq m \\
\{\hat{f}_m, [[\hat{f}_{m-1}, \hat{f}_m]_q, \hat{f}_{m+1}]_{\bar{q}}\} = \{\hat{f}_m, [[\hat{f}_{m-1}, \hat{f}_m]_{\bar{q}}, \hat{f}_{m+1}]_q\} = 0.
\end{gathered} \tag{52}$$

obtained from the $\hat{e}$-Serre relations by replacing every $\hat{e}_k$ with $\hat{f}_k$.

Setting (see (3)–(5))

$$\begin{aligned}
&h_i = \hat{h}_i \quad e_i = \hat{e}_i \qquad i = 1,\ldots,m \\
&h_i = -\hat{h}_i \quad e_i = -\hat{e}_i \qquad i = m+1,\ldots,m+n-1 \\
&f_i = \hat{f}_i \qquad i = 1,\ldots,m+n-1
\end{aligned} \tag{53}$$

one verifies that the defining relations of $U_q[sl(m|n)]$ (50)–(52) are among the defining relations of $U_q[osp(2n+1|2m)]$ (3)–(5). Therefore $U_q[sl(m|n)]$ is a subalgebra of $U_q[osp(2n+1|2m)]$ in a sense of associative algebras.

A set of Cartan–Weyl elements of $U_q[sl(m|n)]$ has been considered in [17], and consists of $m+n-1$ 'Cartan' elements $\tilde{H}_i = e_{11} - (-1)^{\theta_{i+1}} e_{i+1,i+1} = \hat{h}_1 + (-1)^{\theta_2}\hat{h}_2 + (-1)^{\theta_3}\hat{h}_3 + \cdots + (-1)^{\theta_i}\hat{h}_i$ and $(m+n)(m+n-1)$ root vectors e_{ij}, $i \neq j = 1, \ldots, m+n$ ($\hat{e}_i = e_{i,i+1}$, $\hat{f}_i = e_{i+1,i}$). e_{ij} is positive if $i<j$ and negative if $i>j$. Among the positive root vectors, the normal order is given by

$$e_{ij} < e_{kl} \qquad \text{if} \quad i<k \quad \text{or} \quad i=k \quad \text{and} \quad j<l. \tag{54}$$

For the negative root vectors e_{ij} one takes the same rule (54), and the total order is fixed by choosing

$$\text{positive root vectors} \; < \; \text{negative root vectors} < \tilde{H}_i.$$

A complete set of relations between the Cartan–Weyl elements is given by

$$[\tilde{H}_i, \tilde{H}_j] = 0 \tag{55a}$$

$$[\tilde{H}_i, e_{jk}] = (\delta_{1j} - \delta_{1k} - (-1)^{\theta_{i+1}}(\delta_{i+1,j} - \delta_{i+1,k}))e_{jk}. \tag{55b}$$

For any $e_{ij} > 0$ and $e_{kl} < 0$,

$$\begin{aligned}
[\![e_{ij}, e_{kl}]\!] = {} & ((q-\bar{q})\theta(j>k>i>l)(-1)^{\theta_k} e_{kj} e_{il} \\
& - \delta_{il}\theta(j>k)(-1)^{\theta_{kl}} e_{kj} + \delta_{jk}\theta(i>l)e_{il})\tilde{L}_{i-1}\bar{\tilde{L}}_{k-1} \\
& + \tilde{L}_{j-1}\bar{\tilde{L}}_{l-1}(-(q-\bar{q})\theta(k>j>l>i)(-1)^{\theta_j} e_{il} e_{kj} \\
& - \delta_{il}\theta(k>j)(-1)^{\theta_{ij}} e_{kj} + \delta_{jk}\theta(l>i)e_{il}) \\
& + \frac{\delta_{il}\delta_{jk}}{q-\bar{q}} \left(\tilde{L}_{j-1}^{(-1)^{\theta_i}} \bar{\tilde{L}}_{i-1}^{(-1)^{\theta_i}} - \bar{\tilde{L}}_{j-1}^{(-1)^{\theta_i}} \tilde{L}_{i-1}^{(-1)^{\theta_i}} \right).
\end{aligned} \tag{55c}$$

For $0 < e_{ij} < e_{kl}$,

$$[\![e_{ij}, e_{kl}]\!]_{q^{(-1)^{\theta_j}\delta_{jl} - (-1)^{\theta_j}\delta_{jk} + (-1)^{\theta_i}\delta_{ik}}} = \delta_{jk} e_{il} + (q-\bar{q})(-1)^{\theta_k}\theta(l>j>k>i)e_{kj}e_{il}. \tag{55d}$$

For $0 > e_{ij} > e_{kl}$,

$$[\![e_{ij}, e_{kl}]\!]_{\bar{q}^{(-1)^{\theta_j}\delta_{jl} - (-1)^{\theta_j}\delta_{jk} + (-1)^{\theta_i}\delta_{ik}}} = \delta_{jk} e_{il} - (q-\bar{q})(-1)^{\theta_k}\theta(i>k>j>l)e_{kj}e_{il} \tag{55e}$$

$$(e_{ij})^2 = 0 \qquad \text{if} \quad \theta_{ij} = 1 \tag{55f}$$

where

$$\theta(i_1 > i_2 > \cdots > i_k) = \begin{cases} 1 & \text{if} \quad i_1 > i_2 > \cdots > i_k \\ 0 & \text{otherwise} \end{cases} \qquad \tilde{L}_i = q^{\tilde{H}_i} \quad \bar{\tilde{L}}_i = \tilde{L}_i^{-1}.$$

It is tedious but straightforward to check that the expressions

$$\begin{aligned}
& e_{ij} = \tfrac{1}{2}(-1)^{\langle i\rangle}\bar{q}^{(-1)^{\langle j\rangle}N_j - 1/2}[\![c_i^-, c_j^+]\!] \qquad i<j \\
& e_{ij} = \tfrac{1}{2}(-1)^{\langle i\rangle}[\![c_i^-, c_j^+]\!]q^{(-1)^{\langle i\rangle}N_i - 1/2} \qquad i>j \\
& \tilde{H}_i = -N_1 - (-1)^{\langle i+1\rangle}N_{i+1}
\end{aligned} \tag{56}$$

satisfy relations (55). Thus we have obtained a realization of the algebra $U_q[sl(m|n)]$ in terms of the q-anticommuting deformed Bose and Fermi operators (12) as a subalgebra of $U_q[osp(2n+1|2m)]$.

The realization (56) allows one to consider any Fock module and in particular the irreducible $U_q[osp(2n+1|2m)]$ Fock module $F_{1/k}(n|m)$ as a $U_q[sl(m|n)]$ module and to decompose it into a direct sum of irreducible $U_q[sl(m|n)]$ submodules. From (56) we have for the representatives $\rho(e_{ij})$, $i \neq j$, $\rho(\tilde{H}_i)$ of all Cartan–Weyl generators of $U_q[sl(m|n)]$ in the Fock space $F_{1/k}(n|m)$:

$$
\begin{aligned}
&\rho(e_{ij}) = -(-1)^{\langle i\rangle+\langle j\rangle}\cos(\pi/(2k))\exp(-(-1)^{\langle j\rangle}\mathrm{i}\pi N_j/k)c_j^+c_i^- && i<j\\
&\rho(e_{ij}) = -\cos(\pi/(2k))c_j^+c_i^-\exp((-1)^{\langle i\rangle}\mathrm{i}\pi N_i/k) && i>j \qquad (57)\\
&\rho(\tilde{H}_i) = -N_1-(-1)^{\langle i+1\rangle}N_{i+1}.
\end{aligned}
$$

The matrix elements follow from (46):

$$
\rho(\tilde{H}_j)|r_1,\ldots,r_{m+n}) = -(r_1+(-1)^{\langle j+1\rangle}r_{j+1})|r_1,\ldots,r_{m+n}) \qquad (58a)
$$

$$
\begin{aligned}
\rho(e_{jl})|r_1,\ldots,r_{m+n}) ={}& -(1-(1-\langle l\rangle)r_l)(-1)^{(\langle j\rangle-\langle l\rangle)(r_1+\cdots+r_j)+(1-\langle l\rangle)(r_{j+1}+\cdots+r_{l-1})}\\
&\times\exp(-\mathrm{i}\pi(r_j+\cdots+r_{l-1}+(-1)^{\langle l\rangle}r_l-2\langle l\rangle)/k)\\
&\times\sqrt{\frac{\sin(\pi r_j/k)\,\sin(\pi(r_l+1)/k)}{\sin^2(\pi/k)}}|\ldots,r_j-1,\ldots,r_l+1,\ldots) \qquad j<l
\end{aligned}
\qquad (58b)
$$

$$
\begin{aligned}
\rho(e_{jl})|r_1,\ldots,r_{m+n}) ={}& -(1-(1-\langle l\rangle)r_l)(-1)^{(\langle l\rangle-\langle j\rangle)(r_1+\cdots+r_l)+(1-\langle j\rangle)(r_{l+1}+\cdots+r_{j-1})}\\
&\times\exp(\mathrm{i}\pi(r_l+\cdots+r_{j-1}+(-1)^{\langle j\rangle}r_j)/k)\\
&\times\sqrt{\frac{\sin(\pi r_j/k)\,\sin(\pi(r_l+1)/k)}{\sin^2(\pi/k)}}|\ldots,r_l+1,\ldots,r_j-1,\ldots) \qquad j>l.
\end{aligned}
\qquad (58c)
$$

Obviously, the subspace $F_r(m|n)$ spanned by all vectors (35) with a fixed admissible value of $r_1+\cdots+r_{m+n}=r$ is a $U_q[sl(m|n)]$ submodule. Since the matrix elements in front of $|\ldots,r_j-1,\ldots,r_l+1,\ldots)$ and $|\ldots,r_l+1,\ldots,r_j-1,\ldots)$ in the RHS of (58) are different from zero, if these vectors belong to $F_{1/k}(n|m)$ the $U_q[sl(m|n)]$ module $F_r(m|n)$ is irreducible. Hence the irreducible k^m2^n dimensional $U_q[osp(2n+1|2m)]$ Fock module $F_{1/k}(n|m)$ splits into a direct sum of $mk-m+n+1$ irreducible $U_q[sl(m|n)]$ modules $F_r(m|n)$ characterized by a number r taking integer values from 0 up to $m(k-1)+n$. For different k the modules $F_r(m|n)$ have different dimensions and therefore they carry inequivalent irreducible primitive root of unity representations of $U_q[sl(m|n)]$.

4. Concluding remarks

We introduced a construction for a deformation of the Clifford superalgebra $Cl(n|m)$. As a guide for our construction, we used the property that $Cl(n|m)$ is a homomorphic image of $U[osp(2n+1|2m)]$ and assumed that this property remains unchanged, i.e. the deformed superalgebra $Cl_q(n|m)$ must be a homomorphic image of $U_q[osp(2n+1|2m)]$. The resulting $Cl_q(n|m)$ is an associative superalgebra generated by m deformed bosons and n deformed fermions with the somewhat unusual property that the bosons are odd (i.e. fermionic) generators, whereas the fermions are even (bosonic) generators. A crucial role for determination of the explicit form of the homomorphism φ is played by the realization of $U_q[osp(2n+1|2m)]$ via deformed Green generators.

The Fock space of $Cl_q(n|m)$ with the condition

$$
\left(c_i^+\right)^\dagger = c_i^- \quad (N_i)^\dagger = N_i \qquad i=1,\ldots,m+n
$$

motivated by the requirement that the physical observables be Hermitian operators yields necessarily that the deformed bosons are 'finite dimensional', i.e. they act in finite-dimensional state spaces.

Every $F_{l/k}(n|m)$ is mapped by a homomorphism φ into an irreducible $U_q[osp(2n+1|2m)]$ module, thus defining a sequence of root of unity representations of this quantum superalgebra. We decomposed each $F_{1/k}(n|m)$ into a direct sum of irreducible $U_q[sl(m|n)]$ modules and gave a class of root of 1 irreducible finite-dimensional representations of $U_q[sl(m|n)]$. Finally, we derived the action of all Cartan–Weyl elements e_{ij} on the basis vectors.

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